Murnaghan-Nakayama rule for complete flag variety

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Abstract. Schubert polynomials were introduced by A. Lascoux and M. P. Schützenberg to describe the cohomology ring of complete flag varieties. The famous Schur functions are special cases of Schubert polynomials. In this work we study generalization of Murnaghan-Nakayama rule for Schubert polynomials. We found a symmetric formula for this rule using the Fomin-Kirillov algebra.

Introduction

The cohomology ring of the complete flag variety $H^*(\mathcal{F}\ell_n)$ admits a special linear basis $\{\sigma_w\}$ indexed by permutations on n elements. The elements of the basis are called Schubert classes. For any $u, v \in S_n$, we have

$$\sigma_u \sigma_v = \sum_{w \in S_n} c_{u,v}^w \sigma_w$$

for some $c_{u,v}^w \in \mathbb{R}, u, v, w \in S_n$. The numbers $c_{u,v}^w$ are called the structure constants for $H^*(\mathcal{F}\ell_n)$. By algebro-geometric reasons, the structure constants are always non-negative integers. To provide a combinatorial interpretation for these structure constants is a long standing open problem in algebraic combinatorics. The constants are generalizations of famous Littlewood-Richardson coefficients ([9]), which correspond to the case when both permutations are Grassmannian of the same descent.

Study of cohomology rings of flag varieties started long ago and the first multiplication rule was constructed by D. Monk [12] in 1959. I. N. Bernstein, I. M. Gelfand, S. I. Gelfand [1] and M. Demazure [3] gave a description of the cohomology ring of the complete flag variety $\mathcal{F}\ell_n$ in 70th. Later in 1982 A. Lascoux and M. P. Schützenberg [7, 8] defined Schubert polynomials recursively using divided differences operators. For the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, \ldots]$, the i-th divided differences operator is given

by
$$\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}.$$
It is easy to see that these operators send polynomials to polynomials, furthermore, if f has integer coefficients, then $\partial_i f$ also has integer coefficients.

Definition 1 (c.f. [7, 8]). *For a permutation $w_0 = (n, n-1, \ldots, 1) \in S_n$, its Schubert polynomial is given by*
$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 \in \mathbb{Q}[x_1, x_2, \ldots].$$
For a permutation $w \in S_n$ s.t. $w \neq w_0$,
$$\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i} \quad \text{for } i \text{ such that } \ell(ws_i) = \ell(w) + 1.$$

These polynomials are well defined and the definition above agrees with the following inclusion $S_1 \subset S_2 \subset S_3 \subset \ldots \subset S_{\mathbb{N}}$.

Theorem 1 (c.f. [7, 8]). *For any $u \in S_{\mathbb{N}}$, its Schubert polynomial $\mathfrak{S}_u$ is well defined and $\mathfrak{S}_u$ is a homogeneous polynomial of degree $\ell(u)$.*

The set $\{\mathfrak{S}_u, u \in S_{\mathbb{N}}\}$ of all Schubert Polynomials is a linear basis of $\mathbb{Q}[x_1, x_2, x_3, \ldots]$.

The closed formula for each Schubert polynomials in terms of the reduced decompositions was given by S. Billey, W. Jockusch, and R. Stanley [2] and using rc-graphs by S. Fomin and A. N. Kirillov [4], see also [6]. Schubert polynomials are generalizations of famous Schur functions, see the book [10].

Since $\{\mathfrak{S}_u, u \in S_{\mathbb{N}}\}$ is a linear basis of $\mathbb{Q}[x_1, x_2, x_3, \ldots]$, we have unique coefficients $c_{u,v}^w, u, v, w \in S_{\mathbb{N}}$ such that, for any $u, v \in S_{\mathbb{N}}$,
$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_{\mathbb{N}}} c_{u,v}^w \mathfrak{S}_w.$$
These coefficients $c_{u,v}^w, u, v, w \in S_{\mathbb{N}}$ are exactly the structure constants for flag varieties.

The following rule was proven for the original problem.

Theorem 2 (Monk's rule, c.f. [12]). *For $u \in S_{\mathbb{N}}$ and $k \in \mathbb{N}$, we have*
$$\mathfrak{S}_u \mathfrak{S}_{s_k} = \mathfrak{S}_u \cdot (x_1 + x_2 + \ldots + x_k) = \sum_{a \leq k < b:\ \ell(ut_{a,b}) = \ell(u)+1} \mathfrak{S}_{ut_{a,b}},$$
where $t_{a,b}$ is a transposition of a and b.

Later Pieri's rule and a more general rule for rim hooks were given by F. Sottile in 1996 [14]. K. Mészáros, G. Panova, and A. Postnikov in 2014 [11] rewrote and gave a new prove of the rule for rim hooks (and proved that this way works for hooks with extra square) in terms of Fomin-Kirillov algebra [5]. We will define Fomin-Kirillov algebra and formulate Pieri's rule in the next section. There are also some other rules, but unfortunately they have restrictions on both permutations. A. Morrison and F. Sottile found the Murnaghan-Nakayama rule for Schubert polynomials, see [13] and below we develop Murnaghan-Nakayama rule in Fomin-Kirillow algebra. Our formula has extra symmetries unlike Morrison-Sottile's rule and it is better in sense of Bruhat orders, see Proposition 2.

1. Fomin-Kirillov algebra

Denote by $\mathcal{FK}_{\mathbb{N}}$ the algebra with generators $[i,j]$, where $i \neq j \in \mathbb{N}$ and relations

- $[i,j] = -[j,i]$;
- $[i,j]^2 = 0$;
- $[i,j][k,\ell] = [k,\ell][i,j]$ for distinct i,j,k,ℓ;
- $[i,j][j,k] + [j,k][k,i] + [k,i][i,j] = 0$.

The last equation is know as associate Yang-Baxter equation. The classical $\mathcal{FK}_n$ is generated only by $[i,j], i,j \in [n]$. The Fomin-Kirillov algebra acts on Schubert polynomials (on the cohomology ring) as the following one side operators

$$\mathfrak{S}_w[a,b] = \begin{cases} \mathfrak{S}_{wt_{a,b}} & \text{if } \ell(wt_{a,b}) = \ell(w)+1 \text{ and } a<b, \\ -\mathfrak{S}_{wt_{a,b}} & \text{if } \ell(wt_{a,b}) = \ell(w)+1 \text{ and } a>b, \\ 0 & \text{otherwise.} \end{cases}$$

We define Dunkl elements in $\mathcal{FK}_{\mathbb{N}}$ as

$$\theta_k = -\sum_{i<k}[i,k] + \sum_{j>k}[k,j] = \sum_i [k,i].$$

Dunkl elements are commute pairwise, i.e., $\theta_i\theta_k = \theta_k\theta_i$, see [5]. As corollary of Monk's rule we get

Proposition 1 (c.f. [5]**).** *For any permutation $u \in S_n$ and $k \in \mathbb{N}$, we have*

$$\mathfrak{S}_u x_k = \mathfrak{S}_u \theta_k.$$

Theorem 3 (Pieri's rule [11]**).** *For $u \in S_{\mathbb{N}}$ and $k,m \in \mathbb{N}$, we have*

$$\mathfrak{S}_u \cdot h_k(x_1,x_2,\ldots,x_m) = \mathfrak{S}_u \cdot \left(\sum_{i_1 \le i_2 \le \ldots \le i_k \le m} x_{i_1}x_{i_2}\cdots x_{i_k} \right) = \\ = \sum_{\substack{a_1 \le \ldots \le a_k \le m \\ m<b_1,\ldots,b_k \text{ are distinct}}} \mathfrak{S}_u[a_1b_1][a_2b_2]\cdots[a_kb_k]$$

and

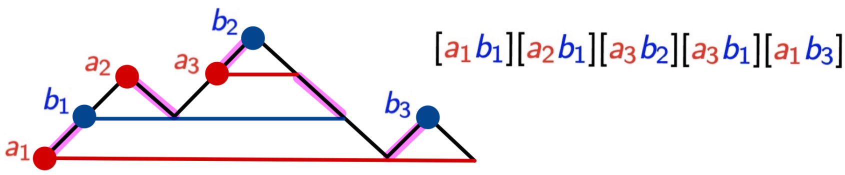
$$\mathfrak{S}_u \cdot e_k(x_1,x_2,\ldots,x_m) = \mathfrak{S}_u \cdot \left(\sum_{i_1 < i_2 < \ldots < i_k \le m} x_{i_1}x_{i_2}\cdots x_{i_k} \right) = \\ = \sum_{\substack{a_1,\ldots,a_k \le m \text{ are distinct} \\ m<b_1 \le \ldots \le b_k}} \mathfrak{S}_u[a_1b_1][a_2b_2]\cdots[a_kb_k].$$

In this paper we extend this approach and present the formula for Murnaghan-Nakayama rule in Fomin-Kirillov algebra.

Theorem 4 (Murnaghan–Nakayama rule). *For $u \in S_{\mathbb{N}}$ and $k, m \in \mathbb{N}$, we have*

$$\mathfrak{S}_u \cdot p_k(x_1, x_2, \ldots, x_m) = \mathfrak{S}_u \cdot (x_1^k + x_2^k + \ldots + x_m^k) =$$
$$= \sum_{P \text{ is a Dyck path of length } 2k} (-1)^{u_e(P)} \sum_{\substack{a_1, \ldots, a_{u_e(P)+1} \leq m \\ b_1, \ldots, b_{k-u_e(P)} > m \\ \text{are distinct}}} \mathfrak{S}_u \mathcal{M}_P(a, b).$$

The first summation is overall Dyck paths and the second summation is overall distinct indexes $a_1, \ldots, a_{u_e(P)+1} \leq m < b_1, \ldots, b_{k-u_e(P)}$ corresponding to moves up on even and odd places resp. and $\mathcal{M}_P(a, b)$ is a product of $[a_i, b_j]$ as in the picture.



It is clear that our rule is symmetric on indexes $[m]$ and on indexes $\{m + 1, m+2, m+3, \ldots\}$, which should help in a construction of such a rule for the case of Schubert polynomials times Schur functions. Our rule is impossible to simplify, see Proposition 2.

Proposition 2. *For $u, v \in S_{\mathbb{N}}$ and $k, m \in \mathbb{N}$, there is at most one Dyck path with indexes $a_1, \ldots, a_{u_e(P)+1} \leq m < b_1, \ldots, b_{k-u_e(P)}$ such that $\mathfrak{S}_u \mathcal{M}_P(a, b) = \mathfrak{S}_v$.*

In particular, $\mathfrak{S}_u p_k(x_1, x_2, \ldots, x_m) = \sum \pm \mathfrak{S}_v$, where summation by some permutations.

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